PARAMETRIZED VECTOR FIELDS AND THE ZERO-CURVATURE CONDITION

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Abstract. We apply the notion of a parametrized vector field on a manifold M, where the parameters are also in M, to the study of the zero-curvature condition that arises in the context of integrable systems.

1. Motivation

Consider the following result, which is perhaps well-known to experts. Let R be a ring of characteristic zero, equipped with a derivation ∂, and let

$$L = \partial + \sum_{i=-\infty}^{-1} f_i \partial^i \tag{1.1}$$

be a first-order monic formal pseudodifferential operator with coefficients in R. Let Ψ_L denote the ring of pseudodifferential operators commuting with L. Given positive integers n, m, there is a Lie algebra homomorphism

$$\Psi_L \to \Psi_L[[s,t]] \tag{1.2}$$

$$K \mapsto K(s,t) \tag{1.3}$$

given by solving the initial value problem

$$\frac{\partial K}{\partial s} = [L_+^n, K] \tag{1.4}$$

$$\frac{\partial K}{\partial t} = [L_+^m, K] \tag{1.5}$$

$$K(0,0) = K\ . \tag{1.6}$$

In particular, flows (1.4) and (1.5) commute. This appears paradoxical at first, since L_+^n and L_+^m do not, in general, commute. The resolution of the paradox is that L_+^n and L_+^m are not fixed operators, but are themselves subject to the flows (1.4) and (1.5).

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Better known is the special case $K = L$, $R = \mathbb{C}[[x]]$. Indeed, one then has the family of KP flows, whose commutativity may be understood from various points of view. ([A, SW]).

Here is a somewhat sharper statement, which may not be so well-known: Let L_1 and L_2 be commuting pseudodifferential operators. Let Ψ_{L_1,L_2} denote the ring of pseudodifferential operators commuting with L_1 and L_2. Given positive integers n, m, there is a Lie algebra homomorphism

$$\Psi_{L_1,L_2} \to \Psi_{L_1,L_2}[[s,t]] \tag{1.7}$$

$$K \mapsto K(s,t) \tag{1.8}$$

given by solving the initial value problem

$$\frac{\partial K}{\partial s} = [L_{1+}, K] \tag{1.9}$$

$$\frac{\partial K}{\partial t} = [L_{2+}, K] \tag{1.10}$$

$$K(0,0) = K\ . \tag{1.11}$$

The assumption that K commutes with L_1 and L_2 is not fundamental. Its role is to guarantee that the order of K remains bounded. Issues of this sort arise in the infinite dimensional setting. Indeed, the commutativity of the flows (1.9) and (1.10) holds with Ψ_{L_1,L_2} replaced by an arbitrary finite dimensional Lie algebra. (Corollary 4.2.)

The purpose of this note is to give an understanding of the commutativity of the flows (1.9) and (1.10) in terms of parametrized vector fields on a manifold, where the parameters are themselves elements of the manifold. Given such a vector field, and given a choice of the parameters, one obtains a flow, with the understanding that the parameters themselves also flow. The next section describes this setup. We specialize to Lie algebras in the subsequent section. To avoid irrelevant complications, we will work with finite dimensional manifolds. Throughout the paper, M denotes such a manifold.

2. N-pvf 's

Definition 2.1. *Given a positive integer n, let $\mathrm{pvf}_n(M)$ denote the space of sections*

$$\xi : M^{n+1} \to p_{n+1}^*(TM)\ , \tag{2.1}$$

where p_i is projection onto the i^{th} factor and TM is the tangent bundle.

We refer to the elements of $\mathrm{pvf}_n(M)$ as n-pvf 's.

One may think of an n-pvf as an object which, for every choice of $a_1, ..., a_n \in M$ determines a flow on M in the following way. Given $b \in M$, one has a tangent vector $\xi(a_1, ..., a_n, b)$ to M at b. Move infinitesimally in the direction of that vector, to a nearby point b'. Having done this for all b, one has done it in particular for $a_1, ..., a_n$, so at the next iteration, move in the direction $\xi(a'_1, ..., a'_n, b')$.

To put this more precisely, let $\mathrm{vf}(M^n)$ denote the space of vector fields on M^n. For $i = 1, ..., n$, set

$$\tau_i : M^n \to M^n \tag{2.2}$$

$$\tau_i(a) = (a_1, ..., a_{n-1}, a_i) \ , \tag{2.3}$$

where $a = (a_1, ..., a_n)$. Taking account of the natural isomorphism

$$T(M^n) = \bigoplus_i {p_i}^*(TM) \ , \tag{2.4}$$

define

$$\mathrm{pvf}_{n-1}(M) \xrightarrow{\beta} \mathrm{vf}(M^n) \tag{2.5}$$

$$\xi \mapsto \tilde{\xi} \ , \tag{2.6}$$

by

$$\tilde{\xi}(a_1, ..., a_n) = (\xi \circ \tau_1(a), ..., \xi \circ \tau_n(a)) \ . \tag{2.7}$$

Note that β has a left inverse,

$$\mathrm{vf}(M^n) \xrightarrow{\pi_n} \mathrm{pvf}_{n-1}(M) \tag{2.8}$$

given by projection onto the n^{th} summand in equation (2.4).

Proposition 2.2. *β maps $\mathrm{pvf}_{n-1}(M)$ isomorphically onto the Lie subalgebra*

$$\{ \ \eta \in \mathrm{vf}(M^n) \mid \forall i \ \tau_i^* \circ \eta = \eta \circ \tau_i^* \ \} \ .$$

Thus $\mathrm{pvf}_{n-1}(M)$ forms a Lie algebra, with bracket

$$[\xi, \psi] = \pi_n([\tilde{\xi}, \tilde{\psi}]) \ . \tag{2.9}$$

Proof. Note that if $k \neq n$,

$$\tau_i^* \circ p_k^* = p_k^* \tag{2.10}$$

$$\tau_k \tau_i = \tau_k \tag{2.11}$$

while

$$\tau_i^* \circ p_n^* = p_i^* \tag{2.12}$$

$$\tau_n \tau_i = \tau_i \ . \tag{2.13}$$

Let $\xi \in \mathrm{pvf}_{n-1}(M)$. Let $f \in C^\infty(M)$. Then for all k,
$$\tilde{\xi} \circ p_k^*(f)|_a = \xi|_{\tau_k(a)}(f) \ . \tag{2.14}$$
If $k \neq n$,
$$\begin{aligned}\tilde{\xi} \circ \tau_i^* \circ p_k^*(f)|_a = \xi|_{\tau_k(a)}(f) = \\ \xi|_{\tau_k\tau_i(a)}(f) = \tau_i^* \circ \tilde{\xi} \circ p_k^*(f)|_a\end{aligned}$$
while
$$\begin{aligned}\tilde{\xi} \circ \tau_i^* \circ p_n^*(f)|_a = \xi|_{\tau_i(a)}(f) = \\ \xi|_{\tau_n\tau_i(a)}(f) = \tau_i^* \circ \tilde{\xi} \circ p_n^*(f)|_a \ .\end{aligned}$$
Thus, $\tau_i^* \circ \tilde{\xi} = \tilde{\xi} \circ \tau_i^*$ for all i.

Conversely, let η be a vector field such that $\tau_i^* \circ \eta = \eta \circ \tau_i^*$ holds for all i. Let η_n denote the n^{th} component of η. Then
$$\begin{aligned}\eta \circ p_i^*(f)|_a = \eta \circ \tau_i^* \circ p_n^*(f)|_a = \\ \tau_i^* \circ \eta \circ p_n^*(f)|_a = \eta_n|_{\tau_i(a)}(f) \\ = \widetilde{\pi_n(\eta)} \circ p_i^*(f)|_a \ .\end{aligned}$$
Thus $\eta = \widetilde{\pi_n(\eta)}$. □

Example 2.3. If $M = \mathbb{R}$, then a 1-pvf is simply a function $f(x, y)$. Then
$$\tilde{f} = f(x,x)\frac{\partial}{\partial x} + f(x,y)\frac{\partial}{\partial y} \tag{2.15}$$
Then the bracket of 1-pvf 's is given by
$$[f, g] = f(x,x)\frac{\partial g}{\partial x} + f(x,y)\frac{\partial g}{\partial y} - g(x,x)\frac{\partial f}{\partial x} - g(x,y)\frac{\partial f}{\partial y} \tag{2.16}$$

Now let ξ be a 1-pvf . Introduce the following 2-pvf 's:
$$\xi^1(a, b, c) = \xi(a, c) \tag{2.17}$$
$$\xi^2(a, b, c) = \xi(b, c) \tag{2.18}$$

Definition 2.4 (Zero-curvature condition). *Fix a 1-pvf , ξ, on M. A pair of points $a, b \in M$ satisfies the zero-curvature condition (for ξ), zcc, if, for all $y \in M$,*
$$\Phi_s^1\Phi_t^2(a, b, y) = \Phi_t^2\Phi_s^1(a, b, y) \ , \tag{2.19}$$
where $\Phi^i_\cdot$ is the flow of $\widetilde{\xi^i}$.

In other words, (a, b) satisfies zcc if the flow determined by b commutes with the flow determined by a.

2.1. **Infinitesimal criterion.** Let $\mathfrak{F}_n$ denote the free Lie algebra on n letters $\alpha_1, ..., \alpha_n$. Let $\mathfrak{I}_n \subset \mathfrak{F}_n$ denote the commutator ideal. Given $w \in \mathfrak{I}_n$, and given elements $x_1, ..., x_n$ in a Lie algebra $\mathfrak{g}$, let $w(x_1, ..., x_n) \in \mathfrak{g}$ denote the element obtained by evaluating α_i at x_i.

For a vector field X on M, denote by Φ^X the one-parameter group of diffeomorphisms generated by X.

Lemma 2.5. *Let X and Y be smooth vector fields on M, and let $p \in M$. If there is a neighborhood $(0,0) \in U \subset \mathbb{R}^2$ such that for all $(s,t) \in U$,*

$$\Phi_t^Y \Phi_s^X(p) = \Phi_s^X \Phi_t^Y(p) \ , \tag{2.20}$$

then

$$\forall w \in \mathfrak{I}_2 \ , \ \ w(X,Y)(p) = 0 \ . \tag{2.21}$$

Conversely, if X and Y are analytic vector fields on an analytic manifold M, then (2.21) *implies* (2.20)*.*

Proof. Assume (2.20). Let

$$U \xrightarrow{\alpha} M \tag{2.22}$$

$$\alpha(s,t) = \Phi_s^X \Phi_t^Y(p) \ . \tag{2.23}$$

From the two sides of (2.20) respectively, one has $\dfrac{\partial \alpha}{\partial t} = Y|_\alpha$ and $\dfrac{\partial \alpha}{\partial s} = X|_\alpha$. It follows that if X and Y are linearly independent at p, one may choose coordinates $(x_1, ..., x_n)$ centered at p, such that

$$X = \frac{\partial}{\partial x_1} + \tilde{X} \tag{2.24}$$

$$Y = \frac{\partial}{\partial x_2} + \tilde{Y} \ , \tag{2.25}$$

where $\tilde{X}$ and $\tilde{Y}$ vanish along $x_2 = x_3 = ... = x_n = 0$. Then (2.21) holds. It is also clear that (2.21) holds if both X and Y vanish at p.

It remains to consider the case that X is nonvanishing in a neighborhood of p, and $Y(p)$ is a multiple of $X(p)$. Note that equation (2.20) holds with p replaced by $\alpha(s,t)$. Therefore, if there is a sequence $s_k \to 0$, such that $X(\alpha(s_k, 0))$ and $Y(\alpha(s_k, 0))$ are linearly independent, (2.21) will hold at p, by continuity. The remaining possibility is that Y is a multiple of X at $\alpha(s,0)$ for all s in a neighborhood of 0. Then

we may assume

$$X = \frac{\partial}{\partial x_1} \tag{2.26}$$

$$Y = f(x_1, ..., x_n)\frac{\partial}{\partial x_1} + \tilde{Y} \ , \tag{2.27}$$

where $\tilde{Y}$ vanish along $x_2 = x_3 = ... = x_n = 0$. This reduces to the case that $M = \mathbb{R}^1$, $X = \dfrac{d}{dx}$, $Y = f(x)\dfrac{d}{dx}$ and $p = 0$. Let $\gamma(t)$ be the integral curve of Y with $\gamma(0) = 0$. Then

$$\Phi^Y_{t_1}(\gamma(t_2)) = \Phi^Y_{t_1}\Phi^Y_{t_2}(0) = \gamma(t_1 + t_2) \ . \tag{2.28}$$

Now

$$\Phi^Y_{t_1}\Phi^X_{\gamma(t_2)}(0) = \Phi^Y_{t_1}(\gamma(t_2)) = \gamma(t_1 + t_2) \tag{2.29}$$

$$= \Phi_{\gamma(t_2)}\Phi^Y_{t_1}(0) = \Phi^X_{\gamma(t_2)}(\gamma(t_1)) = \gamma(t_1) + \gamma(t_2) \ . \tag{2.30}$$

Then there exists a constant, c, such that $Y = c\dfrac{d}{dx}$, so (2.21) holds.

Conversely, in the analytic setting, the Baker-Campbell-Hausdorff formula furnishes a set of elements $w_{i,j} \in \mathfrak{I}_2$ with the following property: Given any analytic function f in a neighborhood of p,

$$f(\Phi^X_{-s}\Phi^Y_{-t}\Phi^X_s\Phi^Y_t(p)) = \sum_{k=0}^{\infty}\frac{1}{k!}(\sum_{i,j>0} s^i t^j w_{i,j}(X,Y))^k(f)|_p \ . \tag{2.31}$$

Thus (2.21) implies (2.20) □

Thus one has the following infinitesimal criterion.

Corollary 2.6. *Fix an analytic 1-*pvf ξ *on an analytic manifold* M. *Then zcc holds for* (a, b) *if and only if, for all* $w \in \mathfrak{I}_2$, $w(\xi^1, \xi^2)(a, b, z) = 0$ *for all* $z \in M$.

3. The case of a Lie algebra

Let $\mathfrak{g}$ be a finite dimensional Lie algebra. An n-pvf on $\mathfrak{g}$ is simply a smooth function $\xi : \mathfrak{g}^{n+1} \to \mathfrak{g}$. In particular, for all smooth $f : \mathfrak{g}^n \to \mathfrak{g}$, consider the n-pvf

$$\xi_f(x_1, ..., x_n, y) = [f(x_1, ..., x_n), y] \ . \tag{3.1}$$

We will say that such an n-pvf is of *Lax type.*

Proposition 3.1. *The* n-pvf *'s of Lax type form a lie subalgebra of the* n-pvf *s. More precisely, set* $x = (x_1, ..., x_n)$ *and set* $[f(x), x] = ([f(x), x_1], ..., [f(x), x_n])$. *Then*

$$[\xi_f, \xi_g] = \xi_{[f,g]'} \ , \tag{3.2}$$

where

$$[f, g]'(x) = dg_x([f(x), x]) - df_x([g(x), x] + [g(x), f(x)] \ . \tag{3.3}$$

Proof.

$$[\xi_f, \xi_g](x, y) = \frac{d}{dt}|_0([g(x + t[f(x), x]), y + t[f(x), y]] - (f \leftrightarrow g)) \ . \tag{3.4}$$

Then use the Jacobi identity. □

Proposition 3.2. *Equation* (3.3) *endows* $C^\infty(\mathfrak{g}^n; \mathfrak{g})$ *with the structure of a Lie algebra.*

Proof. It is not difficult to prove the proposition by direct calculation. A more conceptual proof is as follows. Given a manifold M equipped with an infinitesimal $\mathfrak{g}$-action, $\nabla : \mathfrak{g} \to \mathrm{Der}(C^\infty(M))$, the space $C^\infty(M) \otimes \mathfrak{g}$ has a natural Lie algebra structure, $[\cdot, \cdot]'$, given by

$$[a \otimes X, b \otimes Y]' = a\nabla_X(b) \otimes Y - b\nabla_Y(a) \otimes X + ab \otimes [X, Y] \ . \tag{3.5}$$

There is a natural infinitesimal action of $\mathfrak{g}$ on $\mathbb{C}^\infty(\mathfrak{g}^n)$, induced by the coadjoint action. The resulting bracket is precisely (3.3), up to sign. □

4. Main Result

Note that $C^\infty(\mathfrak{g}^n, \mathfrak{g})$ now has two Lie algebra structures, the pointwise bracket and the bracket given by (3.3) . Denote these two Lie algebras by $C^\infty(\mathfrak{g}^n, \mathfrak{g})^P$ and $C^\infty(\mathfrak{g}^n, \mathfrak{g})'$ respectively.

Though the next theorem shares the hypothesis of the AKS theorem, [A], it seems not to be a corollary.

Theorem 4.1. *Let* $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ *be a vector space direct sum decomposition of* $\mathfrak{g}$, *such that* $\mathfrak{g}_\pm$ *are Lie subalgebras. Let* $\mathfrak{G}^P \subset C^\infty(\mathfrak{g}^n, \mathfrak{g})^P$ *denote the Lie subalgebra generated by the projections* $p_1, ..., p_n$. *Let* $\mathfrak{G}' \subset C^\infty(\mathfrak{g}^n, \mathfrak{g})'$ *denote the Lie subalgebra generated by* $p_{1+}, ..., p_{n+}$, *where* $(\cdot)_+$ *denotes projection onto* $\mathfrak{g}_+$. *Then the map of vector spaces*

$$\begin{aligned} C^\infty(\mathfrak{g}^n, \mathfrak{g}) &\to C^\infty(\mathfrak{g}^n, \mathfrak{g}) \\ f &\mapsto f_+ \end{aligned} \tag{4.1}$$

restricts to a Lie algebra homomorphism

$$\mathfrak{G}^P \to \mathfrak{G}' \ . \tag{4.2}$$

Proof. As before, let $\mathfrak{F}_n$ denote the free Lie algebra on n letters $\alpha_1, ..., \alpha_n$. We must prove that for all $\omega \in \mathfrak{F}_n$,

$$\omega(p_1, ..., p_n)_+ = \omega(p_{1+}, ..., p_{n+}) \ . \tag{4.3}$$

Given a word $w = b_1 b_2 ... b_k$ over the alphabet $\alpha_1, ..., \alpha_n$, let

$$\tilde{w} = [b_1, [b_2, [...[b_{k-1}, b_k]]...] \ . \tag{4.4}$$

The words $\tilde{w}$ span $\mathfrak{F}_n$, so it suffices to establish (4.3) when $\omega = \tilde{w}$. Fix an index i, and consider the set of functions $f \in C^\infty(\mathfrak{g}^n, \mathfrak{g})$ satisfying the following equation:

$$df([p_{i+}, p]) = [p_{i+}, f] \ , \tag{4.5}$$

where $p = (p_1, ..., p_n)$. It is clear that this set of functions forms a Lie subalgebra of $C^\infty(\mathfrak{g}^n, \mathfrak{g})^P$. Furthermore, for all j, (4.5) holds when $f = p_j$. Thus, (4.5) holds for all $f \in \mathfrak{G}^P$. Given (4.5), one has

$$\begin{aligned}
[p_{i+}, f_+]' &= df_+([p_{i+}, p]) - p_{i+}([f_+, p]) + [f_+, p_{i+}] \\
&= [p_{i+}, f]_+ - [f_+, p_i]_+ - [p_{i+}, f_+] \\
&= [p_{i+}, f_-]_+ - [f_+, p_i]_+ \\
&= [p_i, f_-]_+ - [f_+, p_i]_+ \\
&= [p, f]_+ \ .
\end{aligned} \tag{4.6}$$

Then (4.3) holds for all words, by induction on the length.

□

As a corollary, one finds that when a and b are commuting elements of $\mathfrak{g}$, the flows

$$\frac{\partial c}{\partial s} = [a_+, c] \tag{4.7}$$

$$\frac{\partial c}{\partial t} = [b_+, c] \tag{4.8}$$

$$\tag{4.9}$$

commute, irrespective of any functional dependence among a, b and c. This is made precise in the following corollary.

Corollary 4.2. *With $\mathfrak{g}$ as in theorem 4.1, consider the 1-*pvf

$$\xi(x, y) = [x_+, y] \ . \tag{4.10}$$

Then a pair of elements $(a, b) \in \mathfrak{g} \times \mathfrak{g}$ satisfies zcc is and only if, for all $w \in \mathcal{I}_2$, $w(a, b)_+$ belongs to the center of $\mathfrak{g}$.

In particular, for a and b to satisfy zcc it is sufficient that a and b commute.

Proof. By corollary 2.6 , the necessary and sufficient condition is that for all $w \in \mathcal{I}_2$ and all $z \in \mathfrak{g}$, $w(\xi^1, \xi^2)(a, b, z) = 0$. Now ξ^1 and ξ^2 are the Lax type 2-pvf 's $\xi_{p_{1+}}$ and $\xi_{p_{2+}}$ respectively. Then, by proposition 3.1 and theorem 4.1

$$w(\xi^1, \xi^2) = \xi_{w(p_{1+}, p_{2+})} = \xi_{w(p_1, p_2)_+} \ . \tag{4.11}$$

□

This corollary recovers the commutativity of the flows (1.9) and (1.10).

We conclude with a finite dimensional example.

5. Example

Let $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{R})$. Let $\mathfrak{g}_+$ be the subalgebra of skew-symmetric matrices and let $\mathfrak{g}_-$ be the subalgebra of upper triangular matrices. Take

$$a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \ ; \ b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{5.1}$$

First we solve the initial value problem

$$\frac{da(s)}{ds} = [a(s)_+, a(s)] \tag{5.2}$$

$$a(0) = a \ . \tag{5.3}$$

By developing the solution in a power series about $s = 0$, one finds the following solution.

$$a(s) = \begin{bmatrix} -\frac{s}{1+s^2} & 0 & -\frac{s^2}{1+s^2} \\ 0 & 0 & 0 \\ \frac{1}{1+s^2} & 0 & \frac{s}{1+s^2} \end{bmatrix} \tag{5.4}$$

Now the problem

$$\frac{db(s)}{ds} = [a(s)_+, b(s)] \tag{5.5}$$

$$b(0) = b \ . \tag{5.6}$$

can be solved by the dressing method. That is, we let $\sigma(s) \in SL(3, \mathbb{C}[[s]])$ be the solution of the initial value problem

$$\frac{d\sigma(s)}{ds}\sigma(s)^{-1} = a(s)_+ \tag{5.7}$$

$$\sigma(0) = I\ . \tag{5.8}$$

Then

$$b(s) = ad_{\sigma(s)}(b)\ . \tag{5.9}$$

It is clear that $\sigma(s)$ is of the following form:

$$\sigma(s) = \begin{bmatrix} u & 0 & -v \\ 0 & 1 & 0 \\ v & 0 & u \end{bmatrix}\ . \tag{5.10}$$

Then one readily finds $u = \frac{1}{\sqrt{1+s^2}}$ and $v = \frac{s}{\sqrt{1+s^2}}$.

This gives

$$b(s) = \begin{bmatrix} 0 & -\frac{s}{\sqrt{1+s^2}} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{1+s^2}} & 0 \end{bmatrix}\ . \tag{5.11}$$

Next, solve

$$\frac{\partial b(s,t)}{\partial t} = [b(s,t)_+, b(s,t)] \tag{5.12}$$

$$b(s,0) = b(s)\ . \tag{5.13}$$

From looking at the power series expansion of the solution, one is led to make a guess of the following form:

$$b(s,t) = \begin{bmatrix} 0 & X(t) & Y(t) \\ 0 & -\frac{t}{1+s^2+t^2} & -\frac{t^2}{\sqrt{1+s^2}(1+s^2+t^2)} \\ 0 & \frac{\sqrt{1+s^2}}{1+s^2+t^2} & \frac{t}{1+s^2+t^2} \end{bmatrix} \tag{5.14}$$

One finds the following solution:

$$b(s,t) = \begin{bmatrix} 0 & -\frac{s}{\sqrt{1+s^2+t^2}} & -\frac{st}{\sqrt{1+s^2}\sqrt{1+s^2+t^2}} \\ 0 & -\frac{t}{1+s^2+t^2} & -\frac{t^2}{\sqrt{1+s^2}(1+s^2+t^2)} \\ 0 & \frac{\sqrt{1+s^2}}{1+s^2+t^2} & \frac{t}{1+s^2+t^2} \end{bmatrix} \tag{5.15}$$

Finally, one obtains $a(s,t)$ in the form

$$a(s,t) = ad_{\tau(s,t)}(a(s))\ , \tag{5.16}$$

where $\tau(s,t)$ satisfies

$$\frac{d\tau(s,t)}{dt}\tau(s,t)^{-1} = b(s,t)_+ \tag{5.17}$$

$$\tau(s,0) = \sigma(s)\ . \tag{5.18}$$

Here is τ:

$$\tau(s,t) = \begin{bmatrix} \frac{1}{\sqrt{1+s^2}} & 0 & -\frac{s}{\sqrt{1+s^2}} \\ -\frac{s\,t}{\sqrt{1+s^2}\sqrt{1+s^2+t^2}} & \frac{\sqrt{1+s^2}}{\sqrt{1+s^2+t^2}} & -\frac{t}{\sqrt{1+s^2}\sqrt{1+s^2+t^2}} \\ \frac{s}{\sqrt{1+s^2+t^2}} & \frac{t}{\sqrt{1+s^2+t^2}} & \frac{1}{\sqrt{1+s^2+t^2}} \end{bmatrix} \tag{5.19}$$

Then

$$a(s,t) = \begin{bmatrix} -\frac{s}{1+s^2} & \frac{s^2t}{(1+s^2)\sqrt{1+s^2+t^2}} & -\frac{s^2}{\sqrt{1+s^2}\sqrt{1+s^2+t^2}} \\ -\frac{t}{(1+s^2)\sqrt{1+s^2+t^2}} & \frac{t^2s}{(1+s^2+t^2)(1+s^2)} & -\frac{ts}{(1+s^2+t^2)\sqrt{1+s^2}} \\ \frac{1}{\sqrt{1+s^2}\sqrt{1+s^2+t^2}} & -\frac{ts}{(1+s^2+t^2)\sqrt{1+s^2}} & \frac{s}{1+s^2+t^2} \end{bmatrix} \tag{5.20}$$

Finally, for all $c \in \mathfrak{sl}_3(\mathbb{R})$, the solution to

$$\frac{\partial c(s,t)}{\partial s} = [a(s,t)_+, c(s,t)] \tag{5.21}$$

$$\frac{\partial c(s,t)}{\partial t} = [b(s,t)_+, c(s,t)] \tag{5.22}$$

$$c(0,0) = c \tag{5.23}$$

is given by

$$c(s,t) = ad_{\tau(s,t)} ad_{\sigma(s)}(c)\ . \tag{5.24}$$

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